The Impact of a Right of First Refusal Clause in a First-Price Auction with Unknown Heterogeneous Risk-Aversion

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Abstract

We consider that a seller wishes to sell an asset by means of a first-price auction to two potential buyers. One of them may be granted a Right of First Refusal (ROFR) clause which gives him the opportunity to match the bid of the other buyer. When both buyers exhibit heterogeneous risk-attitudes and do not exactly know the competitor’s degree of risk-aversion, we first determine strategic bidding behaviors with and without the ROFR clause. Then, we show that granting the least risk-averse bidder a ROFR can increase the seller’s expected profit. Moreover, we show that the ROFR grows the expected joint profit of the seller and the right-holder but always decreases the non-right-holder expected utility. Finally, we determine the conditions under which the ROFR clause can increase the seller’s expected profit in an open modified auction.

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1 Introduction

The Right of First Refusal (hereafter ROFR) is a right given to a person to be the first person allowed to purchase an asset at the highest price submitted by a third party. The ROFR is commonly used in many contracts: in sports contracts (Employment contracts, especially those of athletes and trainers, may empower the current employer with the ROFR as encouragement to support those unproven talents early in their careers (Chouinard (2005)); in many National football/rugby league ((NFL), the incumbent team has the right to match the best offer a player has once he is eligible to change teams (Lee 2008)); in broadcasting rights, where the incumbent TV channel can match the best bid from another channel; in real estate sales (In France, the law protects the tenant by granting him a ROFR in the sale of a property); in monopoly concession rights (Chouinard (2005)). Hence, different assets may be encumbered by rights of first refusal: commercial assets, corporate securities... Public institutions may also benefit from a ROFR. In 2005, e.g. , the "Dutreil law" created, in France, a ROFR by local municipality, for artisan funds, business assets, commercial leases and lands subject to commercial development projects. The first objective is to protect rural development. In French art auctions, a public institution like The Louvre benefits from a ROFR.

In private procurement auctions, a buyer may also give a preferential treatment to an incumbent supplier. He may namely grant him a ROFR, i.e. a contract clause that provides him with the right to procure an input at the lowest price the buyer is able to get from another supplier, e.g. by a procurement auction. It gives the right-holder the possibility to win the contract by matching the best offer in the first-price reverse auction organized between his rivals.

Is the seller of an asset better off under the ROFR clause? What are the incentives to grant ex-ante a ROFR clause? What are the implications for the right-holder and for other potential competitors? These are the main questions analyzed in this paper. Thus, we consider that a seller wishes to sell an asset to two potential buyers by means of a first-price sealed-bid auction with or without a ROFR clause. We also analyze the impact of a ROFR clause on seller’s expected profit in an open modified auction.
The impact of the ROFR clause seems to be particularly interesting when there are few potential competitors. Indeed, it is important for a seller to find the best selling mechanism to stimulate higher bids when competition is low. Most papers on this subject show that the ROFR clause is better for the favored right-holder but not always for the seller\(^1\) without any legal or illegal side-payments from the right-holder. The ROFR may also increase the risk of an inefficient allocation. As described by Lee (2008), in procurement, motivation for granting a ROFR is often political to simply reward long term business partners. Within a single-object private-value first-price sealed-bid auction, with symmetric and risk-neutral bidders, Arozamena and Wieselthaler (2009) show that a ROFR cannot increase the seller’s expected profit in "regular" cases. However, it increases the "colluded" expected surplus of the seller and the right-holder bidder - while generating a negative externality on all other bidders. Their results are similar to Burguet and Perry (2009) and Choi (2009). In the setting of a first-price auction of National Park Concession contracts, Chouinard (2005) concludes that the suppression of the ROFR increases the service provided by concessioners. Assuming correlation in bidders’ values in a second-price auction, Bikhchandani et al. (2005) show that a ROFR never benefits the seller without any side-payments from the right-holder and that their joint surplus may rise or fall. However, some papers show that the ROFR may increase the seller’s expected profit from the auction. Indeed, in an asymmetric first-price auction with two bidders, Lee (2008) shows that the seller prefers to grant the ROFR to the \(ex\ ante\) weak bidder and that granting this right can benefit the seller. Indeed, the ROFR gives the strong bidder incentive to elicit more aggressive bids than in a standard first-price auction and thus decreases his original advantage.\(^2\) In a symmetric independent private values (IPV) procurement first-price auction, Elmaghraby et al. (2013) show that in a single auction setting, the ROFR does not benefit the seller. However, they show that with a second auction in the future (with the same participating bidders), the seller can increase his profit by granting the ROFR to a bidder in the first of the two sequential auctions. The intuition is that the non-right-holder is induced to bid

\(^1\)This may explain why Nestlé (which is the second biggest shareholder of L’Oréal) announced in 2014 that he would not extend the ROFR clause in his corporate contract with L’Oreal. Nestlé preferred to be free to sell his shares to any potential buyer.

\(^2\)The context of an asymmetric FPA with two bidders is also analyzed by Burguet and Perry (2007) in a model which allows for bribery. See also Piccione and Tan (1996) and Waehrer and Perry (2003) who develop asymmetric auctions models.
very aggressively in the first auction (with ROFR) so that the seller can increase his profit compared to a case of running both sequential auctions without ROFR. So, in a single auction setting, to the best of our knowledge, only the context of an asymmetric auction (described by Lee (2008)) can make the seller better off with the ROFR procedure.\(^3\)

All preceding papers assume that parties are all risk-neutral. However, in practice, in the sale of an asset, e.g., a corporate security, potential buyers may exhibit different attitudes towards risk. In this particular sale, e.g., a firm wishing to become a new minority shareholder in the industry may be more risk-averse than the other shareholders. The ROFR clause also often exists in contracts between a franchisor and a franchisee. If the franchisee decides to sell his business, the franchisor may enforce the clause. We can imagine that a potential competing brand to this franchise may want to purchase the business unit in order to increase its market power in a given geographic area. Then, this external buyer may exhibit more risk-aversion related to this acquisition than the current franchisor. In this paper, we depart from previous papers insofar as we consider that bidders may have heterogeneous degrees of risk-aversion under the assumption of a Constant Relative Risk-Averse (CRRA) model.\(^4\) We also assume that each bidder privately knows his own degree of risk-aversion. Then, we show that the fact that bidders may exhibit different private degrees of risk-aversion provides a new argument for the current value of a ROFR clause that may be enforced at some future time (in a first-price auction). Indeed, we show that when the non-right-holder bidder is highly risk-averse while the right-holder bidder is weakly risk-averse, the ROFR clause can increase the seller’s ex ante expected profit without any side-payments from the right-holder. We also show that the right-holder always benefits from this clause whatever the degree of risk-aversion of the non-right-holder bidder. However, this non-right-holder bidder never benefits from the ROFR clause. We also prove that the ROFR increases the ex ante expected

\(^3\)Burguet and Perry (2007) and Choi (2009) have shown that while granting an ROFR for free never benefits the auctioneer, he may benefit if he sells the ROFR clause before the auction to the bidder with the highest willingness to pay. Hua (2007) also shows that an ex ante ROFR clause may improve social welfare.

\(^4\)Our paper is related to the fundamental experimental paper of Cox et al. (1982) which tests bidding behaviors when bidders exhibit different degrees of constant relative risk-aversion. They show that empirical results support important features of the CRRA model with heterogeneous risk preferences.
joint profit of the seller and the right-holder whatever the non-right-holder’s degree of risk-aversion. Finally, we show that a ROFR clause can also benefit the seller in an open modified auction when the non right-holder with the highest valuation is rather risk-averse and when competition is rather low.

The paper is organized as follows. In the next section, we outline the model and derive bidding strategies under the first-price auction (FPA) mechanism. Section 3 deals with the bidding strategies under the ROFR clause. Thus, we analyze how a ROFR impacts the non-right-holder’s bidding strategy. Section 4 offers a comparison of bidders’ expected utilities and expected joint profits of the seller and the right-holder with and without the ROFR clause. Section 5 deals with the comparison of seller’s expected profits under both procedures (in a first-price auction). In section 6, we analyse the impact of a ROFR clause in an open modified auction. Section 7 offers some concluding remarks.

2  First-price auction with two bidders

Consider a first-price sealed-bid auction with two buyers (bidders, here) $i = 1, 2$ and one asset to be auctioned by a seller. In a standard first-price auction (FPA), the winner is the highest bidder. He pays an amount equal to his bid to the seller. Without loss of generality and to simplify, we consider that the seller’s reservation price is equal to 0. The seller is assumed to be risk-neutral. Each bidder’s monetary valuation $v_i$ for the item is an independent private value from the cumulative distribution function $F(v)$ on $[0, 1]$. $F(v)$ has a continuous probability density function $f(v)$ that is positive on $[0, 1]$. Each bidder $i$ privately knows $v_i$. The distribution and support from which valuations $v_i$ are drawn are common knowledge. Assume that bidder $i$’s preferences over income $y_i$ can be represented by $U_i(y_i) = (y_i)^{r_i}$, where $1 - r_i$ is the CRRA parameter of bidder $i$. $r_1$ and $r_2$ are assumed to be private information for each bidder. More precisely, each bidder doesn’t know the identity of his competitor and only knows that his competitor has a degree of risk aversion $r_1$ with a discrete probability $p$ and $r_2$ with the complement $(1 - p)$.\footnote{See Cox et al. (1982) for an analysis of unknown risk aversion where the risk-aversion parameter is a continuous random variable.} Without loss of generality, assume that $0 < r_2 \leq r_1 \leq 1$. Thus, bidder 2 is more risk-averse than bidder 1. Although
each bidder doesn’t know the identity of his competitor we assume that the seller knows the identity of both bidders.

In our context of heterogeneous risk-aversion levels, let us now determine the bidding strategies of both bidders. First dealing with bidder 1 (the least risk-averse bidder), we can consider that he believes that his rival (bidder 2) uses a bid function $b_2(v_2)$ strictly increasing in $v_2$ with $b_2(0) = 0$ if his degree of risk-aversion is $r_2$ and a symmetric increasing strategy $b_1(v_2)$ if his degree of risk aversion is $r_1$ (as bidder 1) with $b_1(0) = 0$. Thus, the expected utility of bidder 1, when he submits a bid $b_1$, can be written as

$$EU_1(v_1, b_1) = (v_1 - b_1)^{r_1} (p \Pr(b_1(v_2) \leq b_1) + (1-p) \Pr(b_2(v_2) \leq b_1))$$

$$= (v_1 - b_1)^{r_1} (p \Pr(v_2 \leq b_1^{-1}(b_1)) + (1-p) \Pr(v_2 \leq b_2^{-1}(b_1)))$$

$$= (v_1 - b_1)^{r_1} (pF(\phi_1(b_1)) + (1-p) F(\phi_2(b_1)))$$

$$= (v_1 - b_1)^{r_1} G(b_1)$$

where $\phi_i$ is the inverse function of bidder $i$’s strategy when his own degree of risk-aversion is $r_i$ and $G(b_1)$ is the probability that $b_1$ is the winning bid.

The expected utility of bidder 2 is then

$$EU_2(v_2, b_2) = (v_2 - b_2)^{r_2} G(b_2)$$

Assume that the cumulative function $F$ is uniform on $[0,1]$. Then, the following proposition summarizes the bidding strategies\(^6\) of both bidders

**Proposition 1**

(i) Bidder’s 1 optimal strategy is

$$b_1(v_1) = \frac{v_1}{1 + r_1} \text{ with a maximal bid } \overline{b} = \frac{1}{1 + r_1}$$

(ii) Bidder’s 2 optimal strategy is separated into two segments

$$b_{2\text{down}}(v_2) = \frac{v_2}{1 + r_2} \text{ for } v_2 \leq \tilde{v}_2 = \frac{1 + r_2}{1 + r_1}$$

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\(^6\)See the Appendix for a proof.
and

\[ b_{2}^{up}(v_{2}) = \frac{v_{2}}{1 + r_{2}} - \frac{pr_{2}}{(1 - p)(1 + r_{2})} \left[ 1 - \left( \frac{1 + E(r)}{(1 + r_{1})(p + (1 - p)v_{2})} \right)^{\frac{1}{r_{2}}} \right] \text{ for } \tilde{v}_{2} \leq v_{2} \leq 1 \]

The bidding strategies of both bidders are depicted by Figure 1 (for \( p = 0.8, r_{1} = 0.9 \) and \( r_{2} = 0.1 \)).

![Figure 1: Optimal bidding strategies](image)

Notice that bidder 1’s optimal strategy is a linear function of \( v_{1} \). Besides, it only depends on his own degree of risk-aversion \( r_{1} \). More precisely, his bid decreases with \( r_{1} \) and increases with \( v_{1} \) with a maximal bid (i.e. when \( v_{1} = 1 \) \( \bar{b} \)). Concerning bidder 2’s optimal strategy \( b_{2} \), it appears to be kinked. Namely, it is a linear increasing function of \( v_{2} \) for \( v_{2} \leq \tilde{v}_{2} \) (such as \( b_{2}^{down}(\tilde{v}_{2}) = \bar{b} \)) which does not depend on \( p \) and \( r_{1} \), the parameters which capture bidder 1’s risk-aversion. However, for \( v_{2} \geq \tilde{v}_{2} \), it does depend on \( p \) and \( r_{1} \).\(^7\) More precisely, proposition 2 exhibits the properties of bidder 2’s optimal strategy.\(^8\)

\(^7\)Note that, contrary to Cox (1982), the context of a discrete probability distribution allows us to find the equilibrium bidding strategies of both bidders under the FPA when bidders’ valuations are uniformly distributed.

\(^8\)See the Appendix for a proof.
Proposition 2  \textit{Bidder 2's optimal strategy exhibits the following properties}

(i) \( b_{2}^{up} \) is decreasing in \( r_{1} \).

(ii) \( b_{2}^{up} \) is decreasing in \( p \).

(iii) \( b_{2}^{up} \) is a convex increasing function of \( v_{2} \).

(iv) \( b_{2} \) is decreasing in \( r_{2} \).

Why does the most risk-averse bidder (i.e. bidder 2) have a more aggressive bidding strategy than bidder 1? Notice that under the threshold \( \tilde{v}_{2} \), bidder 2's strategy decreases with \( r_{2} \), does not depend on \( r_{1} \) and is more aggressive than bidder 1's strategy as it is generally the case in symmetric risk-aversion cases: More risk-aversion leads to more aggressive bidding (see, e.g., Krishna 2002).\(^{9}\) Although bidder 1's maximal possible bid is \( \bar{b} \), bidder 2 can still have an incentive to bid above \( \bar{b} \). Indeed, bidding above \( \bar{b} \) can increase his probability of winning the auction since he competes, with probability \( (1 - p) \), with another bidder with the same degree of risk-aversion than him. As shown by proposition 2, \( b_{2}^{up} \) decreases with \( p \) and with \( r_{1} \) and is always lower than \( \frac{v_{2}}{1 + r_{2}} \). Indeed, the higher the probability \( p \), the lower the probability of competing with a bidder with risk-aversion \( r_{2} \). And the higher \( r_{1} \), the lower \( \bar{b} \) is, which gives bidder 2 an incentive to decrease his bidding strategy above \( \tilde{v}_{2} \).

3  \textit{Comparison of the non-right-holder’s bidding strategies with and without the ROFR clause}

Consider now that bidder 1 was granted a ROFR ex ante, i.e. before he knows his own valuation. We can either assume that the seller and bidder 1 signed this clause in a private context or that the ROFR clause was required by law in order e.g. to favor public institutions. Thus, bidder 1 no longer takes part in the auction. We analyze here the bidding behavior of bidder 2 (the non-right-holder) when each bidder knows his own valuation and only the cumulative distribution function \( F(v) \) of

\(^{9}\)A more aggressive bidder puts more weight to the probability of winning compared to the profit when he wins.
his competitor. Under this "ROFR procedure", the sequence of events is as follows.
Bidder 2 first submits his bid. Then, bidder 1 is given the opportunity to match bidder 2’s submitted bid. The best decision for bidder 1 is to accept to buy the asset if the bid submitted by bidder 2 is lower than his own valuation $v_1$. If the ROFR clause is not used by bidder 1, the seller then awards the asset to bidder 2 at a price equal to his bid. So, bidder 2 wins against bidder 1 if his bid $b_2^{rofr}$ is higher than $v_1$, which occurs with probability $F(b_2^{rofr})$. Thus, for bidder 2, bidder 1’s strategy can be interpreted as bidding his own valuation $v_1$.\textsuperscript{10} As previously shown by Burguet and Perry (2007), in this setting, the game is dominance solvable and bidder 2’s expected utility under the ROFR procedure is

\[
EU_2(\hat{b}_2^{rofr}) = (v_2 - b_2^{rofr})^r_2 \Pr(v_1 \leq b_2^{rofr}) = (v_2 - b_2^{rofr})^r_2 F(b_2^{rofr})
\]

The first-order condition for expected utility maximization yields

\[
\hat{b}_2^{rofr} = v_2 - r_2 \frac{F(b_2^{rofr})}{f(b_2^{rofr})}
\]

When $F(.)$ is uniform on $[0, 1]$, we have

\[
\hat{b}_2^{rofr}(v_2) = \frac{v_2}{1 + r_2} \text{ for all } v_2 \in [0, 1]
\]

We can now compare bidder 2’s equilibrium bidding strategies under the FPA and the ROFR procedures. Then, the following proposition can be stated\textsuperscript{11}

\textsuperscript{10}In this paper, we assume that the right-holder does not have to submit a bid but only decides whether or not to match his opponent’s bid. The assumption is without loss of generality, as the the right-holder would have a weakly dominant strategy of placing a bid equal to 0 (as in Lee(2008)).

\textsuperscript{11}See the Appendix for a proof.
Proposition 3

(i) When \( r_1 = r_2 \), bidder 2 submits the same bid under both procedures \((b_2^{rofr}(v_2) = b_2^{down}(v_2) = \frac{v_2}{1+r_2})\).

(ii) When \( r_1 > r_2 \) and \( v_2 \leq \tilde{v}_2 \) (with \( \tilde{v}_2 = \frac{1+r_2}{1+r_1} \)), bidder 2 submits the same bid under both procedures \((b_2^{rofr}(v_2) = b_2^{down}(v_2) = \frac{v_2}{1+r_2})\).

(iii) When \( r_1 > r_2 \) and \( v_2 > \tilde{v}_2 \), bidder 2 is more aggressive under the ROFR procedure than under the FPA \((b_2^{rofr}(v_2) > b_2^{up}(v_2))\).

(iv) When \( r_1 > r_2 \), \( b_2^{up}(v_2) \to b_2^{rofr}(v_2) \) when \( r_2 \to 0 \).

Notice that when both bidders exhibit the same degree of risk-aversion (i.e. \( r_1 = r_2 \)), bidder 2 has the same strategy under both procedures. As previously shown by Porter and Shoham (2005) and Arozamena and Weinschelbaum (2009), under risk-neutrality and symmetric distributions of valuations, this result only holds for specific distributions of valuations. Indeed, they show that when \( F(\cdot)/f(\cdot) \) is linear (which is the case for a uniform distribution) and bidders are risk-neutral, bidder 2, facing a rival who holds a ROFR, should behave just as under a FPA. This result may seem to be counter-intuitive but the interpretation given by Arozamena and Weinschelbaum (2009) is the following. When bidder 1 is granted a ROFR, does bidder 2 has an incentive to bid more aggressively under the ROFR procedure than under a two-bidder FPA? Actually, bidder 2 knows that bidder 1 is ready to be more aggressive under the ROFR procedure than under the FPA since bidder 1 is ready to increase his bid to his own valuation. Thus, since the probability of winning of bidder 2 is reduced, he has an incentive to bid more aggressively. However, there is a counteracting effect. The marginal behavior of bidder 1 is less aggressive under the ROFR than under the FPA. So, when e.g. bidder 2 becomes more aggressive, the impact on his marginal probability of winning is stronger under the FPA than under the ROFR. This change in the marginal behavior of bidder 1 provides bidder 2 with incentives to become less aggressive. In the special case where \( F(\cdot)/f(\cdot) \) is

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12 Under a symmetric risk-neutral setting, Arozamena and Weinschelbaum (2009) and Porter and Shoham (2004) have shown that convexity (or concavity) of the inverse hazard rate of bidders’ distributions of valuations is the condition that determines whether non-right-holder bidders should bid more (or less) aggressively when a bidder is granted a ROFR.
linear, both effects exactly offset one another.

Under the ROFR procedure, remark that the strategic behavior of the right-holder does not depend on his risk-aversion level, since his optimal strategy is to enforce the ROFR clause as long as his valuation for the asset is higher than his competitor’s bid. Then, assume that the ROFR was granted to the most risk-averse bidder. In this case, the least risk-averse bidder (the non-right-holder) would not modify his bid compared to the FPA and so would not be strategically more aggressive in the ROFR than in the FPA. So, obviously, we have the following corollary

**Corollary 1** *Under the ROFR procedure, the seller is better off when the least risk-averse bidder holds the ROFR.*

So, in this paper, we always consider that the ROFR is granted to bidder 1 (the least risk-averse bidder).

In an asymmetric first-price auction with two bidders whose valuations are drawn from different distributions, Lee (2008) shows that the auctioneer might prefer to grant a ROFR to the weak bidder, so as to induce the strong bidder to bid more aggressively. In view of Lee’s finding, our result that the auctioneer might grant a ROFR to the least risk-averse bidder might appear surprising. Without the ROFR, the most risk-averse bidder bids more aggressively than the least risk-averse bidder. Thus, a naive conclusion from Lee could be that the ROFR should be granted to the most risk-averse bidder, so as to induce the least risk-averse bidder to submit a higher bid. Our difference with Lee’s conclusion can be explained by the difference in the nature of asymmetry. In Lee’s model, both bidders’ cumulative functions are from the same support, and asymmetry comes from the size of the support: The highest weak bidder’s valuation is always lower than the highest strong bidder’s valuation. In this setting, under the ROFR procedure, "eliciting more aggressive bids from the strong bidder by favoring his weak opponent yields more gain than vice versa, because the strong bidder has more scope for bidding aggressively". In our setting, both bidders’ valuations are drawn from the same support and asymmetry comes from risk-aversion. Then, it is not possible to give more incentives to the strong bidder (the least risk-averse bidder, i.e. bidder 1) to submit a more aggressive bid in the ROFR procedure than under the FPA. In particular, under our setting, the strong bidder (i.e. bidder 1)’s
strategy under both mechanisms does not depend on the weak bidder’s risk-aversion level. In contrast, the weak bidder has more incentive to bid more aggressively under the ROFR procedure than under the FPA above the threshold $\tilde{v}_2$. Indeed, bidder 2 knows that, under the FPA, the maximum bid of bidder 1 is $\tilde{b}$. Under the ROFR procedure, for bidder 2 (the non-right-holder), the right-holder’s strategy can be interpreted as bidding his own valuation $v_1$, which may be higher than $\tilde{b}$. Hence, we see that the implications of asymmetries in auctions for the seller are generally difficult to understand and depend on the nature of asymmetries. In particular, it is not clear what Lee’s result means for the case with different risk-aversion levels.

In another asymmetric setting, Burguet and Perry (2007) analyze the impact of favoritism in a first-price procurement auction between a weak bidder and the auctioneer assuming that two bidders have different distributions of valuations on the same support. When the bribe equals 0, the setting can be interpreted as a ROFR procedure: Favoritism means that the favored supplier who looses the bidding (or does not bid) can still obtain the contract at a price equal to the lowest bid from the other suppliers, but without paying a bribe to the auctioneer (or the buyer). The authors analyze the impact of favoritism on bidding strategies. However, in their setting, it is difficult to solve the equilibrium bidding strategy in the FPA without favoritism. It is easier in the case of favoritism because of dominance solvable (the right-holder strategy always consists in "bidding" his own valuation). It can also be shown that their asymmetric setting in the FPA with neutral bidders is equivalent to a situation where bidders have different CRRA utilities but exactly know their rival’s degree of risk-aversion.\(^{13}\) In our setting where each bidder only knows the discrete probability distribution of his rival’s degree of risk-aversion and so may have a different degree, equilibrium conditions are different and we have succeeded in solving this equilibrium strategy under the FPA when bidder’s valuations are uniformly distributed.

\(^{13}\)See Krishna (2002). We are grateful to a referee for pointing out this equivalence.
4 Comparison of bidders’ expected utilities and expected joint profits

Given bidders’ equilibrium strategies under both procedures, we can analyze the effect of the ROFR clause on bidders’ expected utilities and expected joint profits of the seller and the right-holder. Notice that we compare the FPA and the ROFR procedures when bidders and the seller have symmetric information about the distribution of bidders’ valuations, i.e. before each bidder knows his own valuation for the asset sold.

4.1 Bidders’ expected utilities

Let us first deal with the comparison of expected utilities of bidder 1 (i.e. the right-holder under the ROFR procedure) under both procedures.

**Proposition 4** Bidder 1’s expected utility is larger under the ROFR procedure than under the FPA whatever $r_1$ and $r_2$.

Let us give a proof of this proposition. For a given valuation $v_1$:

- Consider firstly that $v_2 \leq \tilde{v}_2$. Then, bidder 2’s strategy is the same under both procedures. If $0 \leq v_2 \leq b_2^{\text{down}}(b_1(v_1))$, bidder 1 wins in both procedures and pays a lower bid in the ROFR procedure (where he pays $b_2^{\text{down}}(v_2)$ rather than $b_1(v_1)$ in the FPA). If $b_2^{\text{down}}(b_1(v_1)) < v_2$, bidder 1 never wins under the FPA but may win (with a positive profit) under the ROFR procedure if $b_1(v_1) \leq b_2^{\text{down}}(v_2) \leq v_1$.

- Consider secondly that $v_2 \geq \tilde{v}_2$. Then, bidder 1 never wins in the FPA but may win (with a positive profit) under the ROFR procedure when $b_2^{\text{up}}(v_2) \leq v_1$.

Hence, these conditions are sufficient to prove that bidder 1’s expected utility is always higher under the ROFR procedure. □

Let us now deal with bidder 2. The following proposition can be stated.

**Proposition 5** Bidder 2’s expected utility is lower under the ROFR procedure than under the FPA.

Bidder 2 is always negatively impacted by the ROFR procedure. This proposition can be easily proved. Below the threshold $\tilde{v}_2$, bidder 2 submits the same bidding
strategy under both procedures but his probability of winning is larger under the FPA. Above the threshold $\tilde{v}_2$, bidder 2 is more aggressive under the ROFR than under the FPA and his probability of winning is always lower than under the FPA (since bidder 1 never wins in the FPA when $v_2 \geq \tilde{v}_2$). □

4.2 Expected joint profits of the seller and the right-holder

In this section, we deal with the comparison of the expected joint profits of the seller and bidder 1 (the right-holder). Assuming risk-neutrality for bidder 1 allows our results to be compared with preceding papers in the literature where all bidders are risk-neutral. Then, we have the following proposition\(^{14}\)

**Proposition 6** The expected joint profit of the seller and the right-holder is always larger under the ROFR procedure than under the FPA whatever bidder 2’s degree of risk-aversion.

Our result complements those of e.g. Arozemena et al. (2009) and Burguet and Perry (2009) who obtained the same conclusion in a symmetric risk-neutral bidders setting.\(^{15}\) This result explains why the seller and the right-holder can, *ex ante*, have an incentive to negotiate a ROFR clause. Let us now deal with the comparison of seller’s expected profits under both procedures.

5 Comparison of seller’s expected profits

Under the FPA, the seller’s expected profit is equal to the expected highest bid between both bidders. Under the ROFR procedure, bidder 1 is given the opportunity to match bidder 2’s submitted bid. So, under the ROFR procedure, whatever the winner, the seller’s expected profit is equal to the expected bid submitted by bidder 2. From proposition 3, when $r_1 > r_2$, bidder 2 is more aggressive under the ROFR procedure than under the FPA when his valuation is higher than $\tilde{v}_2$ (and is as aggressive as under the FPA when his valuation is lower than $\tilde{v}_2$). So, one might expect that the ROFR clause would lead to an increase of the seller’s expected profit. However, this "aggressiveness" effect is balanced by a "competition" effect which is the

\(^{14}\)See the Appendix for a proof.

\(^{15}\)Notice that although these authors do not consider risk-aversion, they deal with a model with $n$ bidders and a much larger class of distributions than we do.
result of the reduced competition in the auction (under the ROFR procedure) since bidder 1 no longer takes part in the auction. Under the FPA, the seller's expected profit is the highest expected bid between both bidders. Under the ROFR procedure, the seller's expected profit is not any more the highest expected bid between both bidders. It is equal to the expected bid submitted by bidder 2. Then, we can consider the two following cases:

- Consider firstly that \( v_2 < \tilde{v}_2 = \frac{1+r_2}{1+r_1} \). If \( b_1(v_1) > b_2^{\text{down}}(v_2) \) i.e. \( v_2 < \frac{v_1(1+r_2)}{1+r_1} \), bidder 1 wins the FPA and the seller's expected profit is \( b_1(v_1) \). Under the ROFR procedure, the seller expected profit would have been equal to \( b_2^{\text{rofr}}(v_2) \). So, the seller's expected profit is higher under the FPA than under the ROFR procedure. If \( v_2 > \frac{v_1(1+r_2)}{1+r_1} \), bidder 2 wins the FPA and the seller's expected profit is the same under both procedures.

- Consider secondly that \( v_2 > \tilde{v}_2 \). Then, bidder 1 can not win the FPA any more. And since bidder 2 is more aggressive under the ROFR procedure than in the FPA, the seller's expected profit is higher under the ROFR procedure.

So, a necessary condition for the ROFR procedure to yield a higher expected profit for the seller than the FPA is \( v_2 > \tilde{v}_2 \). Since \( r_1 > r_2 \), this last condition is obviously more likely to be satisfied when the term \( \frac{1+r_2}{1+r_1} \) is low. Let us now compare seller's expected profits under both procedures. Then, we have

\[
\Delta = ER_{\text{fpa}} - ER_{\text{rofr}} = \int_0^1 \int_0^{(\frac{1+r_2}{1+r_1})v_1} \left( b_1(v_1) - b_2^{\text{rofr}}(v_2) \right) dv_2 dv_1 \\
+ \int_0^{\frac{1+r_2}{1+r_1}} \int_{b_2^{\text{up}}(v_2)}^{b_2^{\text{rofr}}(v_2)} \left( b_2^{\text{up}}(v_2) - b_2^{\text{rofr}}(v_2) \right) dv_2 \\
\text{A}(r_1, r_2) \quad \text{B}(r_1, r_2, p)
\]

The first term \( \text{A}(r_1, r_2) \) corresponds to the competition effect while the second term \( \text{B}(r_1, r_2, p) \) reflects the aggressiveness effect (which only depends on the "upper" part of bidder 2's bidding strategy under the FPA, \( b_2^{\text{up}}(v_2) \)). Obviously, we have \( \text{A}(r_1, r_2) > 0 \) and \( \text{B}(r_1, r_2, p) < 0 \).

For the seller, the choice between FPA and ROFR procedures results from a
trade-off between getting the highest bid from bidder 1 in the FPA (part $A(r_1, r_2)$, competition effect which is always positive) and a more aggressive bid from the non-right-holder in the ROFR (part $B(r_1, r_2, p)$, aggressiveness effect, which is always negative). Then, if $A(r_1, r_2) < (\text{resp. } >) - B(r_1, r_2, p)$, the seller’s expected profit will be higher under the ROFR (resp. FPA) procedure. Concerning the properties of $A$ and $B$, we first have the following lemma

**Lemma 1** $A(r_1, r_2)$ and $B(r_1, r_2, p)$ are decreasing in $r_1$.

Proof of Lemma 1. $A(r_1, r_2)$ is always positive and depends on $r_1$ from the upper bound of the second integral $\left(\frac{1+r_2}{1+r_1}\right)v_1$ which decreases with $r_1$ and from $b_1(v_1)$ which decreases with $r_1$. $B(r_1, r_2, p)$ is always negative and depends on $r_1$ from the lower bound of the second integral $\frac{1+r_2}{1+r_1}$ which decreases with $r_1$ and from $b_2^{up}(v_2)$ which decreases with $r_1$. □

Since $A(r_1, r_2)$ and $B(r_1, r_2, p)$ are decreasing in $r_1$, we have

**Lemma 2** $\Delta$ is decreasing in $r_1$.

Since $b_2^{up}(v_2)$ is decreasing in $p$, we also have the following lemma

**Lemma 3** $\Delta$ is decreasing in $p$.

The following proposition provides a comparison between seller’s expected profits under both procedures and derives some conditions under which one procedure dominates the other.

**Proposition 7**

(i) When $r_1 = r_2$, $ER_{fpa} > ER_{rofr}$.

(ii) When $r_2 \to 0$, $ER_{fpa} > ER_{rofr}$.

(iii) For a given $\tau_2$, if $r_1^*$ and $p^*$ exist such that $A(r_1^*, \tau_2) + B(r_1^*, \tau_2, p^*) < 0$, then we have $A(r_1, \tau_2) + B(r_1, \tau_2, p) < 0$ (or $ER_{fpa} < ER_{rofr}$), $\forall r_1 > r_1^*$ and $\forall p > p^*$.

(iv) If for a given $r_1^*$ and a given $p^*$, we have $A(r_1^*, r_2) + B(r_1^*, r_2, p^*) > 0$, then we have $ER_{fpa} > ER_{rofr}$ $\forall r_1 \leq r_1^*$. 

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(v) If for a given \( r_1^{**} \) and a given \( p^{**} \), we have \( A(r_1^{**}, r_2) + B(r_1^{**}, r_2, p^{**}) > 0 \), then we have \( ER_{fpa} > ER_{rofr} \) \( \forall p \leq p^{**} \).

Proof of proposition 7.

(i) \( B(r_1, r_2, p) = 0 \) and \( A(r_1, r_2) > 0 \).

(ii) \( B(r_1, r_2, p) \to 0 \) since \( b_2^{up}(v_2) \) tends to \( b_2^{rofr}(v_2) \).

(iii) , (iv) and (v). Proofs result directly from Lemma 2 and Lemma 3.

□

The first result (i) is straightforward. From proposition 3, we know that bidder 2 submits the same bid under both mechanisms \( (b_2^{rofr}(v_2) = b_2^{down}(v_2) = \frac{v_2}{1+r_2}) \) when \( r_2 = r_1 \) and for all \( v_2 \in [0,1] \). Under the FPA, the price paid to the seller is the highest bid which comes from the bidder with the highest valuation (since both bidders use symmetric strategies when \( r_2 = r_1 \)). Hence:

- If \( v_1 > v_2 \), the price paid to the seller is \( b_1(v_1) \) under the FPA and it is \( b_2^{rofr}(v_2) \) under the ROFR procedure. In this case, the seller’s expected profit is higher under the FPA than under the ROFR procedure since \( b_1(v_1) > b_2^{rofr}(v_2) \).

- If \( v_2 > v_1 \), the price paid to the seller is \( \frac{v_2}{1+r_2} \) under both procedures.

So, the seller’s expected profit can not be higher under the ROFR procedure than under the FPA.

Consider now the case of heterogeneous risk-averse bidders (with \( r_1 > r_2 \)). When \( r_2 \) tends to 0, the aggressiveness effect tends to 0 (result (ii)) and the seller’s expected profit only depends on the competition effect and so is always higher under the FPA than under the ROFR. Figure 2 illustrates the evolution of \( ER_{fpa} - ER_{rofr} \) (when \( r_1 = 1 \) while \( r_2 \) increases from 0.001 to 0.99 and \( p \) increases from 0.001 to 0.99) and shows that we can have situations where conditions given in (iii) or (iv) and (v) are satisfied. Then, we can have situations where the seller’s expected profit is higher under the ROFR procedure for a given \( r_2 \) when \( p \) is higher than a certain threshold (result (iii)). In other situations, when \( p \) tends to 1 (\( p=0.99 \), for some \( r_2 \), the seller’s expected profit is always higher under the FPA. Then, we can conclude that this last result will be true for all \( r_1 < 1 \) and for all \( p < 0.99 \) (result (iv)).
Given lemma 1, the aggressiveness effect is more likely to be stronger than the competition effect when $r_1 = 1$. Indeed, above a threshold $p^*$, how can we explain that $ER_{fpa} < ER_{rofr}$ between two thresholds $r_2^*(p)$ and $r_2^{**}(p)$? When $r_2 \to 0$, we always have $ER_{fpa} > ER_{rofr}$ since the aggressiveness effect tends to 0. However, for a sufficiently high value of $p$, when $r_2$ increases, the competition effect increases too but the aggressiveness effect decreases at first and then increases with $r_2$ and is null when $r_2 \to 0$ and when $r_2 = r_1$. Then, in some settings, when the aggressiveness effect (which is always negative) decreases with $r_2$, we can have a first threshold $r_2^*(p)$ above which this effect is stronger than the competition effect, even if this last effect always increases with $r_2$. And then, when $r_2$ continues to increase, the aggressiveness effect becomes increasing in $r_2$ (i.e. is less negative), the competition effect continues to increase with $r_2$ and becomes stronger than the aggressiveness effect above the second threshold $r_2^{**}(p)$.

6 Impact of a ROFR in an open modified auction

It is well known that the standard open auction yields a lower expected profit for the seller than the FPA when bidders are risk-averse (since optimal bidding strategies do not depend on bidders’ risk-aversion in the standard open auction). However, the
seller could use an open modified auction, as described by Burguet and Perry (2009), where one bidder is granted a ROFR. Under this procedure, when no other remaining bidders are willing to improve on the last offer of one of them, the last bidder may still make a final offer increasing his price taking into account the competition with the private valuation of the right-holder. Note that Brisset and Naegelen (2006) analyze this optimal strategy in a similar context in which this last bidder makes an offer against the seller’s secret reserve price.

Under an open modified auction where \( n \) bidders compete in the auction and another bidder is granted a ROFR, the optimal strategy of the non right-holder bidder \( i \), with the highest valuation \( v_i(1) \) among \( n \) bidders in the auction, is to make an offer \( b(v_i(1)) = \frac{v_i(1)}{1 + r_i} \) against the right-holder. If this bid is higher than the second highest valuation \( v(2) \) among \( n \) bidders, then the seller’s profit is \( b(v_i(1)) \). If \( b(v_i(1)) < v(2) \), the seller’s profit is \( v(2) \). So, under the open modified auction, the seller’s profit is equal to \( \text{Max}(b(v_i(1)); v(2)) \), where \( v(1) \) is the expected highest valuation among \( n \) bidders who compete in the auction.

Under our context of risk-averse bidders, let us now analyze the impact of a ROFR on the seller’s profit in this open modified auction.

6.1 The two-bidder case

Let us first deal with the two-bidder case. One bidder holds a ROFR and so does not compete in the auction (e.g. \( n = 1 \)). Assume that the degree of risk-aversion of the non right-holder bidder \( i \) is \( r_i \). Then, as already mentionned above, his optimal bid (against the right-holder), which is also the winning price here, is

\[
b(v_i(1)) = \frac{v_i(1)}{1 + r_i}
\]

Note that this optimal bidding strategy depends on \( r_i \) which is not the case in an open auction without a ROFR. Then, in an open modified auction, the seller’s expected profit, \( ER_{oma} \), is
\[ ER_{oma} = \int_{0}^{1} \frac{v^{i}(1)}{1 + r^{i}} dv^{i} \]
\[ = \frac{1}{2(1 + r^{i})} \]

In an open auction without a ROFR, the seller’s expected profit, \( ER_{ooa} \), is equal to the second highest expected valuation among 2 bidders. Thus, we have

\[ ER_{ooa} = \int_{0}^{1} 2v_{2}(1 - v_{2}) dv_{2} \]
\[ = \frac{1}{3} \]

Then, the following proposition can be stated

**Proposition 8** When the degree of risk-aversion of the non right-holder is lower than \( \frac{1}{2} \), the open modified auction (with a ROFR) increases the seller’s expected profit in comparison with an open auction (without a ROFR).

Proof of proposition 8.

\( ER_{oma} > (resp. <) ER_{ooa} \iff r_{i} < (resp. >) \frac{1}{2}. \] □

### 6.2 The \( n + 1 \) bidder case

Let us now deal with the \( (n + 1) \) bidder case.\(^{16}\) In the open auction (without a ROFR) with \( (n + 1) \) bidders, the seller’s expected profit is

\[ ER_{ooa} = \int_{0}^{1} \int_{0}^{v_{(2)n}(n + 1)v_{(2)}^{n-1}} dv_{(1)}dv_{(2)} \]
\[ = \frac{n}{n + 2} \]

\(^{16}\)Note that a comparison of auction procedures with and without a ROFR is possible in the context of the open auction whereas it was not under the context of the FPA, since we did not find analytic solutions for the bidding strategies under the FPA with \( n \) bidders.
Consider now the open modified auction where \( n \geq 2 \) bidders compete in the auction and one bidder is awarded a ROFR. Dealing with a comparison of seller’s expected profits under an open auction with and without a ROFR where \( n \geq 2 \) requires to compute the bidding strategies in both cases. Namely, a non right-holder bidder not only competes with \((n - 1)\) other bidders in the auction but also competes with the right-holders.

So, under the open modified auction, the winning bid is \( \text{Max}(b(v_{(1)}); v_{(2)}) \) which yields the following seller’s expected profit:

\[
ER_{oma} = \frac{1}{n + 1} \int_0^{v_{(1)}} \int_0^{v_{(2)}} \frac{v_{(1)}^n(n - 1)v_{(2)}^{n-2}dv_{(1)}dv_{(2)}}{1 + r_i} \\
+ \frac{1}{n + 1} \int_0^{v_{(1)}} \int_0^{v_{(2)}} v_{(2)}n(n - 1)v_{(2)}^{n-2}dv_{(1)}dv_{(2)}
\]

\[
= \frac{n - 1}{n + 1} + \frac{1}{(1 + n)(1 + r_i)^n}
\]

In this last equation the term \( \frac{n - 1}{n + 1} \) corresponds to the seller’s expected revenue under an open auction with \( n \) bidders. The term \( \frac{1}{(1 + n)(1 + r_i)^n} \) represents the impact of non right-holders’ reaction due to the presence of a right-holder. Note that this term decreases with \( r_i \) and \( n \). Thus, we see that the interest of the open modified auctions decreases when the number of competitors increases or when the non right-holder with the highest valuation exhibits a low risk-aversion level.

Comparing \( ER_{oma} \) and \( ER_{ooa} \) in the \((n+1)\) bidder case, the following proposition can be stated\(^{17}\)

**Proposition 9** A unique threshold value \( r_i^*(n) = \left(\frac{n + 2}{2}\right)^{\frac{1}{n}} - 1 \), decreasing in \( n \), exists such that under this threshold value, \( ER_{oma} > ER_{ooa} \).

Proposition 9 confirms that the seller benefits from granting a ROFR in an open auction when the non right-holder with the highest valuation is rather risk-averse or

\(^{17}\)See the Appendix for a proof.
when competition is rather low. For example, when $n = 3$, $r^*_i(3) = 0.36$ and when $n = 6$, $r^*_i(6) = 0.26$.

7 Conclusion

In this paper, we analyze the impact of a ROFR clause when an asset is sold by an auction with risk-averse buyers. Our model allows us to consider the realistic assumption that each buyer has a private risk-aversion level. Even if we analyze the special case of a uniform distribution for bidders’ valuations, we obtain several results about the ROFR’s economic impact both in the context of a FPA and in the context of an open modified auction.

Firstly, we consider the context of a FPA with two potential risk-averse buyers. To the best of our knowledge, our paper is the first to obtain explicit equilibrium bidding functions under a FPA where each buyer only knows the (discrete) probability distribution of his rival’s degree of risk-aversion. When buyers are equally risk averse, we show that the seller’s ex ante expected profit is larger when there is no ROFR clause. This result is consistent with preceding results of the literature which deals with symmetric risk-neutral bidders in a single auction setting. However, when buyers exhibit heterogeneous risk-attitudes and when the least risk-averse buyer holds a ROFR, we show that the seller’s expected profit can be increased if the non-right-holder buyer is sufficiently risk-averse (but not to much either) while the right-holder is rather risk-neutral. This result is explained by the fact that the non-right-holder buyer is strategically more aggressive in the ROFR procedure than in the FPA when his valuation is higher than a certain threshold. However, if the ROFR clause is granted to the most risk-averse buyer, the ROFR has no impact on the non-right-holder bidder’s strategy. So, in this case, the ROFR does not benefit the seller. We also show than the ROFR always improves the seller and the right-holder’s expected joint profit whatever the non-right-holder’s degree of risk-aversion. This last result explains why future associate members have an incentive to include a ROFR clause in their investment contract. Our model also shows that the ROFR always benefits the right-holder unlike the non-right-holder.

Secondly, we analyze the ROFR’s economic impact in the context of a modified open auction (with $n + 1$ bidders). Then, we show that a ROFR clause can benefit
the seller when the non right-holder bidder with the highest valuation is rather risk-averse or when competition is rather low.

So, when bidders are risk-averse, the ROFR clause can benefit the seller whether the asset is sold via a FPA or an open modified auction. However, in both types of auction, an "unfavored buyer" may have little incentive to participate in a sale if he knows that a bidder has a preferential treatment. This raises the question of the endogenous participation of an unfavored bidder. When entry is endogenous, Jehiel and Lamy (2017) analyze a mechanism with a ROFR which discriminates against a new entrant (with a positive cost of participation) in favor of a right-holder who always participates. They show that this mechanism is not optimal because it reduces the attractiveness of potential entrants, irrespective of their ex ante characteristics and strength.

8 Appendix

Proof of proposition 1

For a bidder $i$, the first-order conditions for expected utility maximisation given that each bidder has the same probability expectations $G(.)$, yields

$$
\phi_i(b_i) = b_i + r_i \frac{G(b_i)}{G'(b_i)} \text{ or } b_i = v_i - r_i \frac{G(b_i)}{G'(b_i)}
$$

Under the assumption that function $\frac{G(b)}{G'(b)}$ is not decreasing, for a given $r_i$, the valuation $v_i = b + r_i \frac{G(b)}{G'(b)}$ is the highest valuation that would generate a bid no greater than $b$. Then, bidder 1, the least risk-averse bidder, would not generate a bid higher than $\tilde{b}$ with $\tilde{b} = 1 - r_1 \frac{G(b)}{G'(b)}$. At equilibrium, bidder 2’s maximal bid is at least $\tilde{b}$. And we note $\tilde{v}_2 = \tilde{b} + r_2 \frac{G(b)}{G'(b)}$, bidder 2’s highest valuation that generates a bid no greater than $\tilde{b}$. When $r_2 < r_1$, we have $r_2 \frac{G(b)}{G'(b)} < r_1 \frac{G(b)}{G'(b)}$, and we can conclude that $\tilde{v}_2 < 1$. So, in the region where $b \leq \tilde{b}$, the probability that each bidder will bid less or equal to $b$ is given by

$$
G(b) = pF(\phi_1(b)) + (1 - p) F(\phi_2(b))
$$
Under a uniform distribution for $F$ and given the FOC (3) for each bidder, we have at equilibrium

$$G(b) = p\phi_1(b) + (1 - p)\phi_2(b) = p(b + r_1 \frac{G(b)}{G'(b)}) + (1 - p)(b + r_2 \frac{G(b)}{G'(b)})$$

(4)

So we have

$$G(b) = b + E(r) \frac{G(b)}{G'(b)} \text{ with } E(r) = pr_1 + (1 - p)r_2$$

(5)

In (5), $E(r)$ denotes the expectation taken on risk-aversion coefficients. Equation (5) is a linear first-order differential function with the initial condition $G(0) = 0$. Note that Cox, Smith and Walker (1982) was the first to show how the solution of the inverse equilibrium function in this asymmetric setting could be obtained and reduced to a single differential equation in $G$ and $b$ that can be directly integrated for $b \leq \bar{b}$. In our simple setting, it is easy to show that the unique solution of this equation, given the initial condition, is linear and given by

$$G(b) = (1 + E(r))b$$

So, we conclude that $\frac{G(b)}{G'(b)} = b$ and given (3), we have for each bidder

$$\phi_1(b) = b + r_1b \text{ and } \phi_2(b) = b + r_2b$$

So, in the region where $b \leq \bar{b}$, i.e. for $v_1 \in [0, 1]$ and $v_2 \in [0, \tilde{v}_2]$, the Nash equilibrium bid functions for bidder 1 and bidder 2 are

$$b_1(v_1) = \frac{v_1}{1 + r_1} \text{ for } v_1 \in [0, 1]$$

$$b_2(v_2) = \frac{v_2}{1 + r_2} \text{ for } v_2 \in [0, \tilde{v}_2]$$

So, we have $\bar{b} = \frac{1}{1 + r_1}$ and $\tilde{v}_2 = \bar{b} + r_2\bar{b} = \frac{1 + r_2}{1 + r_1}$.

Now, above $\bar{b}$, in contrast with the setting first developed by Cox et al. (1982, 1988) and numerically improved by Boening, Rassenti and Smith (1998), in our own setting of a discrete distribution of risk-aversion coefficients, we can obtain a closed form solution for bidder 2’s Nash equilibrium bid function for $b > \bar{b}$. The analysis is the following. First note that a bidder with type $r_1$ (with probability $p$) has no
incentive to bid higher than $\bar{b}$. So, above $\bar{b}$, bidder 2’s probability of winning with a bid $b > \bar{b}$ is

$$G(b) = p + (1 - p) \phi_2(b)$$

(6)

Given that (3) defines the inverse of bidder 2’s equilibrium bid function, we have

$$G(b) = p + (1 - p) \left( b + \frac{r_2 G(b)}{G'(b)} \right)$$

(7)

Note that the function $b_2(v_2)$ is not differentiable at $\bar{b}$. Indeed, there is a knot at $\bar{b}$ which separates $b_2(v_2)$ in two segments.

For $b < \bar{b}$, we have

$$G(\bar{b}) = (1 + E(r)) \bar{b} = p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)}$$

(8)

So, finding the Nash equilibrium bid function $b_2(v_2, r_2)$ above $\bar{v}_2$ consists in finding the solution of the non-linear first-order differential solution (7) with the initial value problem defined in (8). The solution of this problem consists in finding $b$ as a function of $G$. Then, equation (7) can be written as

$$G = p + (1 - p) \left( b + \frac{r_2 G}{G'} \right)$$

(9)

$$= p + (1 - p) \left( b + r_2 G \frac{db}{dG} \right)$$

(10)

So, we have to find $b(G)$ which is the solution of the first-order differential equation

$$\frac{db}{dG} + \frac{1}{r_2 G} b = \frac{G - p}{(1 - p) r_2 G}$$

(11)

with the initial condition on $G$ deduced from (5)

$$b \left( G(\bar{b}) \right) = b \left( p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)} \right) = \bar{b}$$

(12)

which gives

$$b \left( p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)} \right) = \frac{1}{1 + r_1}$$

(13)
Note that the general solution of (11) (without the initial condition) is linear and given by
\[ b(G) = \frac{C}{G^{\frac{1}{r_2}}} - \frac{p(1 + r_2) - G}{(1 - p)(1 + r_2)} \]
where \( C \) is a constant. Given the initial condition (12), we obtain
\[ C = \left( p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)} \right)^{\frac{1}{r_2}} \frac{pr_2}{(1 - p)(1 + r_2)} \]
Finally, we have
\[ b(G) = \frac{(p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)})^{\frac{1}{r_2}} \frac{pr_2}{(1 - p)(1 + r_2)}}{G^{\frac{1}{r_2}}} - \frac{p(1 + r_2) - G}{(1 - p)(1 + r_2)} \]
and as \( G = p + (1 - p)v_2 \), we obtain
\[
b(v_2) = \frac{(p + (1 - p) \frac{(1 + r_2)}{(1 + r_1)})^{\frac{1}{r_2}} \frac{pr_2}{(1 - p)(1 + r_2)}}{(p + (1 - p)v_2)^{\frac{1}{r_2}}} - \frac{p(1 + r_2) - p + (1 - p)v_2}{(1 - p)(1 + r_2)}
= \frac{v_2}{1 + r_2} - \frac{pr_2}{(1 - p)(1 + r_2)} \left[ 1 - \left( \frac{1 + E(r)}{(1 + r_1)(p + (1 - p)v_2)} \right)^{\frac{1}{r_2}} \right]
\text{for } \tilde{v}_2 \leq v_2 \leq 1 \]
\[ \square \]

**Proof of proposition 2**

(i) **Proof that \( b_{up}^2 \) is decreasing in \( r_1 \)**

Let us compute
\[
\frac{\partial b_{up}^2(v_2)}{\partial r_1} = -\frac{p(1 + p(r_1 - r_2) + r_2)^{\frac{1 - r_2}{r_2}} (p + v_2(1 - p))^{\frac{1}{r_2}}}{(1 + r_1)^2}
\]
Since \( r_1 \geq r_2 \), we have obviously \( \frac{\partial b_{up}^2(v_2)}{\partial r_1} \leq 0. \) \[ \square \]
(ii) Proof that $b_2^{up}$ is decreasing in $p$

We can compute

\[
\frac{\partial b_2^{up}(v_2)}{\partial p} = \frac{p(1 + r_1)(1 + p(r_1 - r_2) + r_2) \frac{1}{r_2} - 1 ((1 + r_1)(p + v_2(1 - p))^{1 - \frac{1}{r_2}} (v_2(1 + r_1) - r_2 - 1)}{(1 + r_2)(1 - p)}
\]

\[
+ \frac{r_2 \left( \frac{1 + p(r_1 - r_2) + r_2}{(1 + r_1)(p + v_2(1 - p))} \right)^{\frac{1}{r_2}} - 1}{(1 + r_2)(1 - p)^2}
\]

Denote $A$ as the RHS of this latter equation. Then we have

\[
\frac{\partial A}{\partial v_2} = - \frac{\left( \frac{1 + p(r_1 - r_2) + r_2}{1 + p(r_1 - r_2) + r_2} \right)^{\frac{1}{r_2} - 1}}{r_2(1 + r_1)(p + v_2(1 - p))^2} (v_2 + p(v_2(1 + r_1 - r_2) - 1))
\]

which is of the opposite sign of $B$. Then we have

\[
\frac{\partial B}{\partial v_2} = 1 + p(r_1 - r_2) + r_2 > 0
\]

So, $B$ is increasing in $v_2$.

At $v_2 = \frac{1 + r_2}{1 + r_1}$, we have $B = \frac{r_2(1 + p(r_1 - r_2) + r_2)}{1 + r_1} > 0$, which implies that $B > 0$.

$\forall v_2 > \frac{1 + r_2}{1 + r_1}$. So, we have $\frac{\partial A}{\partial v_2} > 0$ and thus $\frac{\partial b_2^{up}(v_2)}{\partial p}$ is decreasing in $v_2$. At $v_2 = \frac{1 + r_2}{1 + r_1}$, we have $\frac{\partial b_2^{up}(v_2)}{\partial p} = 0$. So $\frac{\partial b_2^{up}(v_2)}{\partial p} < 0 \forall v_2 > \frac{1 + r_2}{1 + r_1}$. □

(iii) Proof that $b_2^{up}$ is increasing and convex in $v_2$

Let us compute

\[
\frac{\partial b_2^{up}(v_2)}{\partial v_2} = \frac{1 - p(\frac{1 + p(r_1 - r_2) + r_2}{1 + r_1})^{\frac{1}{r_2}} (p + v_2(1 - p))^{-\frac{1 + r_2}{r_2}}}{1 + r_2}
\]

and

\[
\frac{\partial^2 b_2^{up}(v_2)}{\partial v_2^2} = \frac{(1 - p)p(\frac{1 + p(r_1 - r_2) + r_2}{1 + r_1})^{\frac{1}{r_2}} (p + v_2(1 - p))^{-\frac{1 + 2r_2}{r_2}}}{r_2}
\]
Obviously $\frac{\partial^2 b_{2}^{up}(v_2)}{\partial v_2^2} \geq 0$. So, $b_{2}^{up}$ is a convex function of $v_2$ and $\frac{\partial b_{2}^{up}(v_2)}{\partial v_2}$ is increasing in $v_2$.

When $v_2 = \tilde{v}_2$, $\frac{\partial b_{2}^{up}(v_2)}{\partial v_2} = \frac{1-p}{1+p(r_1-r_2)+r_2} > 0$. Since $\frac{\partial b_{2}^{up}(v_2)}{\partial v_2}$ is increasing in $v_2$, this implies that $\frac{\partial b_{2}^{up}(v_2)}{\partial v_2} \geq 0$ for $v_2 \geq \tilde{v}_2$. We conclude that $b_{2}^{up}$ is increasing and convex in $v_2$ for all $v_2 \in [\tilde{v}_2, 1]$. □

(iv) Proof that $b_2$ is decreasing in $r_2$

Let us consider two risk-aversion levels $r_2^a$ and $r_2^b$ with $r_2^a < r_2^b$. These risk-aversion levels yield two thresholds $\tilde{v}_2^a = \frac{1+r_2^a}{1+r_1}$ and $\tilde{v}_2^b = \frac{1+r_2^b}{1+r_1}$. Consider a bidder with valuation $v_2$. Then, we have to focus on three different cases:

1. If $v_2 < \tilde{v}_2^a$, bidder 2’s optimal strategy is $b_{2}\downarrow_{2}(v_2, r_2^a)$ when $r_2 = r_2^a$ and becomes $b_{2}\downarrow_{2}(v_2, r_2^b)$ when $r_2 = r_2^b$. Given that $b_{2}\downarrow_{2}(v_2, r_2) = \frac{v_2}{r_2}$, it is straightforward that $b_{2}\downarrow_{2}$ is decreasing in $r_2$.

2. If $v_2 > \tilde{v}_2^b$, bidder 2’s optimal strategy is $b_{2}^{up}(v_2, r_2^b)$ when $r_2 = r_2^b$ and becomes $b_{2}^{up}(v_2, r_2^a)$ when $r_2 = r_2^a$. From (3) we know that these optimal strategies are respectively the solutions of the first-order conditions (for a given $b_2^b$ and $b_2^a$)

\[
v_2 - b_2^b = r_2^b \frac{G(b_2^b)}{G'(b_2^b)}
\]

and

\[
v_2 - b_2^a = r_2^a \frac{G(b_2^a)}{G'(b_2^a)}
\]

Let us now use a proof by contradiction. To begin, assume that

\[
b_2^b = b_{2}^{up}(v_2, r_2^b) > b_2^a = b_{2}^{up}(v_2, r_2^a)
\]  

(14)

Then we have

\[
v_2 - b_2^b = r_2^b \frac{G(b_2^b)}{G'(b_2^b)} > r_2^a \frac{G(b_2^a)}{G'(b_2^a)}
\]  

(15)

Then, there exists a valuation $v_2^*$ such that $b_2^b$ is the optimal solution $b_{2}^{up}(v_2^*, r_2^a)$. Given (15), we obtain

\[
v_2 - b_2^b = r_2^b \frac{G(b_2^b)}{G'(b_2^b)} \sqrt{(v_2^* - b_2^b)^2}
\]  

(16)

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which implies that \( v_2 > v_2^* \). As \( b_2^{up}(v_2, r_2^a) \) is increasing in \( v_2 \), we can conclude that \( b_2^{up}(v_2, r_2^a) > b_2^{up}(v_2, r_2^a) \). This implies that \( b_2^* > b_2^+ \) which is in contradiction with (14).

3. If \( \tilde{v}_2 \leq v_2 \leq \bar{v}_2 \), bidder 2’s optimal strategy is \( b_2^{up}(v_2, r_2^a) \) when \( r_2 = r_2^a \) and becomes \( b_2^{down}(v_2, r_2^b) \) when \( r_2 = r_2^b \). Both functions are strictly increasing in \( v_2 \). So, we have \( b_2^{up}(\tilde{v}_2, r_2^a) = \bar{b} \) and this function is strictly increasing in \( r_2 \). So, \( \forall v_2 > v_2^a \) we have \( b_2^{up}(v_2, r_2^a) = \bar{b} \). We also have \( b_2^{down}(\tilde{v}_2, r_2^b) = \bar{b} \) and as this function is strictly increasing in \( r_2 \), we have \( b_2^{down}(v_2, r_2^b) < \bar{b} \) \( \forall v_2 < \bar{v}_2 \).

So \( \forall v_2 \) such that \( \tilde{v}_2 \leq v_2 \leq \bar{v}_2 \), we have \( b_2^{up}(v_2, r_2^a) > b_2^{down}(v_2, r_2^b) \). We conclude that bidder 2’s optimal strategy is decreasing in \( r_2 \). □

**Proof of proposition 3**

(ii) Under the ROFR procedure, we have \( b_2^{rofr}(v_2) = \frac{v_2}{1+r_2} \) for all \( v_2 \in [0, 1] \). Under the FPA, we have shown that bidder 2’s optimal bidding strategy is \( b_2^{down}(v_2) = \frac{v_2}{1+r_2} \) for \( v_2 \leq \tilde{v}_2 \) and \( b_2^{up}(v_2) \) for \( v_2 \in [\tilde{v}_2, 1] \). Hence, when \( r_1 = r_2 \), we have \( \tilde{v}_2 = 1 \) and bidder 2’s bidding strategies are identical in both procedures for all \( v_2 \in [0, 1] \). □

(iii) For \( v_2 \in [\tilde{v}_2, 1] \), we can compute

\[
b_2^{rofr}(v_2) - b_2^{up}(v_2) = \frac{pr_2}{(1-p)(1+r_2)} \left[ 1 - \left( \frac{p + (1-p)(1+r_2)}{p + (1-p)v_2} \right)^{\frac{1}{r_2}} \right]
\]

Since \( v_2 \geq \frac{1+r_2}{1+r_1} \), we have \( \left( \frac{p + (1-p)(1+r_2)}{p + (1-p)v_2} \right)^{\frac{1}{r_2}} \leq 1 \) which implies that \( b_2^{rofr}(v_2) \geq b_2^{up}(v_2) \). □

**Proof of proposition 6**

Consider firstly that \( v_2 < \tilde{v}_2 \), then we know that bidder 2’s bidding strategies are equal under both procedures i.e. \( b_2^{down}(v_2) = b_2^{rofr}(v_2) \)

- if \( \frac{v_2}{1+r_1} > \frac{v_2}{1+r_2} \) i.e. \( v_2 < \frac{(1+r_2)v_2}{(1+r_1)} \) then bidder 1 wins the FPA and the joint profit under the PFA is

\[
\pi_j^{fpa} = b_1(v_1) + v_1 - b_1(v_1) = v_1
\]
In this case, since \( v_1 > \frac{v_2}{1+r_1} (1 + r_1) \), we have \( v_1 > b_2^{down}(v_2) \) and thus bidder 1 uses the ROFR clause. Then the joint profit under the ROFR procedure is

\[
\pi_{rofr}^j = v_1 - b_2^{down}(v_2) + b_2^{down}(v_2) = v_1
\]

- if \( \frac{v_1}{1+r_1} < \frac{v_2}{1+r_2} \) i.e. \( v_2 > \frac{(1+r_2)v_1}{(1+r_1)} \) then bidder 2 wins the FPA and the joint profit under the FPA is

\[
\pi_{fpa}^j = b_2^{down}(v_2) + 0 = b_2^{down}(v_2)
\]

Under the ROFR procedure

- If \( v_1 > b_2^{rofr}(v_2) \) then bidder 1 uses the ROFR and \( \pi_{rofr}^j = v_1 - b_2^{rofr}(v_2) + b_2^{rofr}(v_2) = v_1 \)
- If \( v_1 < b_2^{rofr}(v_2) \) then bidder 1 do not use the ROFR and \( \pi_{rofr}^j = b_2^{rofr}(v_2) \)

Consider secondly that \( v_2 > \tilde{v}_2 \), then we know that \( b_2^{up}(v_2) < b_2^{rofr}(v_2) \).

Under the FPA, bidder 2 wins and we have \( \pi_{fpa}^j = b_2^{up}(v_2) + 0 = b_2^{up}(v_2) \)

Under the ROFR procedure:

- If \( v_1 > b_2^{rofr}(v_2) \) then bidder 1 uses the ROFR and \( \pi_{rofr}^j = v_1 - b_2^{rofr}(v_2) + b_2^{rofr}(v_2) = v_1 \)
- If \( v_1 < b_2^{rofr}(v_2) \) then bidder 1 do not use the ROFR and \( \pi_{rofr}^j = b_2^{rofr}(v_2) \)

In each cases, it is easy to see that the joint profit under the ROFR procedure is either equal or higher than the joint profit under the FPA. This implies that the expected joint profit is higher under the ROFR procedure than under the FPA. □

**Proof of proposition 9**

\( ER_{oma} \) is decreasing in \( r_i \) and \( ER_{oma} \) does not depend on \( r_i \). Then, if a threshold \( r_i^*(n) \) exists, it is necessarily unique.

Besides, \( ER_{oma} = ER_{ooa} \) when \( r_i^*(n) = \left( \frac{n+2}{2} \right)^{\frac{1}{n}} - 1 \). Let us now show that \( 0 < r_i^*(n) < 1 \).

\( \left( \frac{n+2}{2} \right)^{\frac{1}{n}} \) is decreasing in \( n \). So, \( r_i^*(n) \) is decreasing in \( n \). Since obviously \( \left( \frac{n+2}{2} \right)^{\frac{1}{n}} > 1 \), we have \( r_i^*(n) > 0 \). So, the maximum value of \( \left( \frac{n+2}{2} \right)^{\frac{1}{n}} \) is obtained with \( n = 2 \) \( \forall n \geq 2 \),
and this value is equal to $2^{1/2}$. Then, since $0 < 2^{1/2} - 1 < 1$, we have $r_i^*(n) < 1 \ orall n \geq 2$. Therefore, we have $0 < r_i^*(n) < 1$.
□

References


